COSC252: Programming Languages:

Semantic Specification

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Outline

I. What happens after syntactic analysis (parsing)?
II. Attribute Grammars: bridging the gap
III. Semantic Specifications
    I. Operational Semantics
    II. Denotational Semantics
    III. Axiomatic Semantics

Parsing

- Our heartless / soulless parser is simply a recognizer
  - Identifies whether a sentence is in a language
    - Whether a code file abides by the rules of a programming language

- Parse errors can be identified and described at this stage.

- Observe: Many implementations of parsers, e.g. c++ compiler, can identify errors that are more closely related to semantics (as compared to syntax)
  - How is this done?

Attribute Grammars

- Attribute Grammars are an extended form of a CFG that can account for “other rules” that can be determined statically, but cannot be accounted for using standard CFGs. (Knuth)
  - Compatibility

- Examples (what types of errors can be identified statically, but not with a standard CFG):
  1. Variable not in scope, not accessible
  2. Multiple definitions in same scope
  3. Type incompatibility
     - Example: function returns a float but a Node* is expected
  - These are errors that are not syntactic, but can be recognized statically (before runtime).

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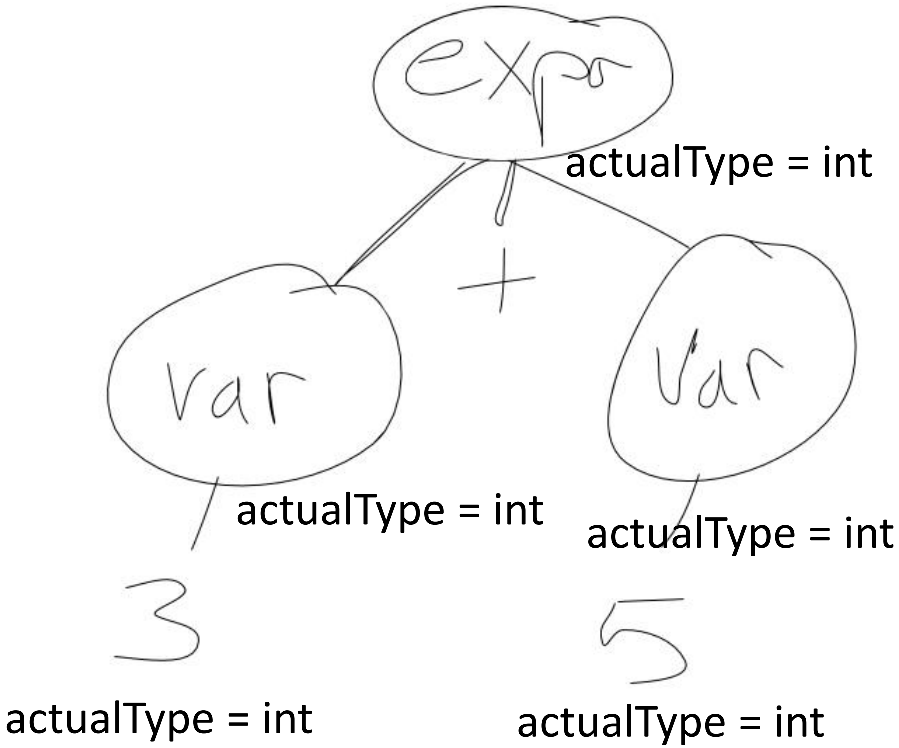
Static Semantics

- Static Semantics are "syntax" rules that are partially related to semantics.
  - "Static" as we can check the rules before runtime, during parsing
- Definitions
  - Attribute is a characteristic of a terminal or non-terminal
  - Semantic Rule Functions are associated with grammar rules
  - Predicate functions: state the static semantic rules associated with a grammar rule
- Attribute Grammar is a CFG with the following:
  - Attributes for a CFG symbol X, A(X)
  - Semantic Rule Function: for each rule in the grammar, $X_0 \rightarrow X_1 X_2 \ldots X_n$ a semantic rule S(X_0) computes the attributes of X_0 given the attributes of $X_1\, X_2 \ldots X_n$, $S(X_0) = f(A(X_0), A(X_1), \ldots, A(X_n))$.
  - Predicate function: is a Boolean expression on the attributes of a grammar. A false value of a predicate function implies that a static semantics rule has been violated.
- A parse tree with an attributed grammar may have attributes, semantic rules, and a predicate function associated with each node.
- If all the attribute values of a parse tree have been computed, the parse tree is said to be fully attributed.
- Intrinsic attributes: are attributes of terminals – leaf nodes in a parse tree

Example: Attribute Grammar

- Attribute grammar to test for compatibility
  - Attributes: expectedType, actualType
  - Grammar:

1. <expr> -> $< num_1 > + < num_2 >$
   1. Semantic Rule:
      if $< num_1 >.actualType == int$ && $< num_2 >.actualType == int$
      then $< expr >.actualType = int$
      else
      then $< expr >.actualType =$ other
   2. Predicate Rule:
      $< expr >.actualType == < expr >.expectedType$
2. <expr> -> <num>
   1. Semantic Rule: $< expr >.actualType = < num >.actualType$
   2. Predicate Rule: $< expr >.actualType == < expr >.expectedType$
3. <num> -> 0 | 1 | …| 9
   1. Semantic Rule: $< var >.actualType = int$



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Example: Attribute Grammar

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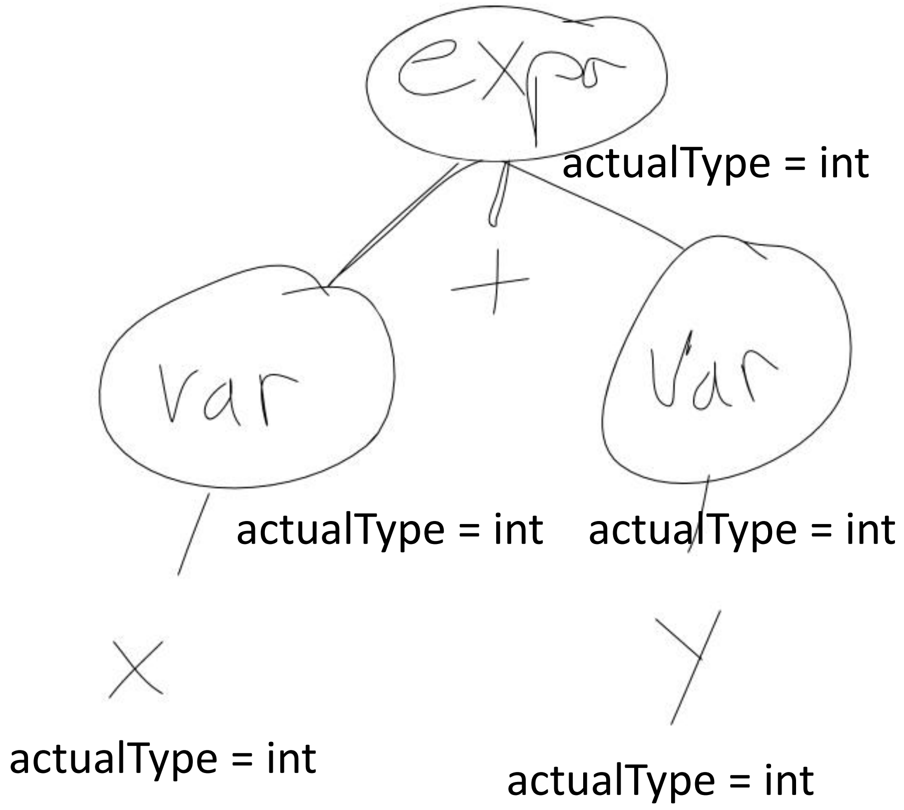
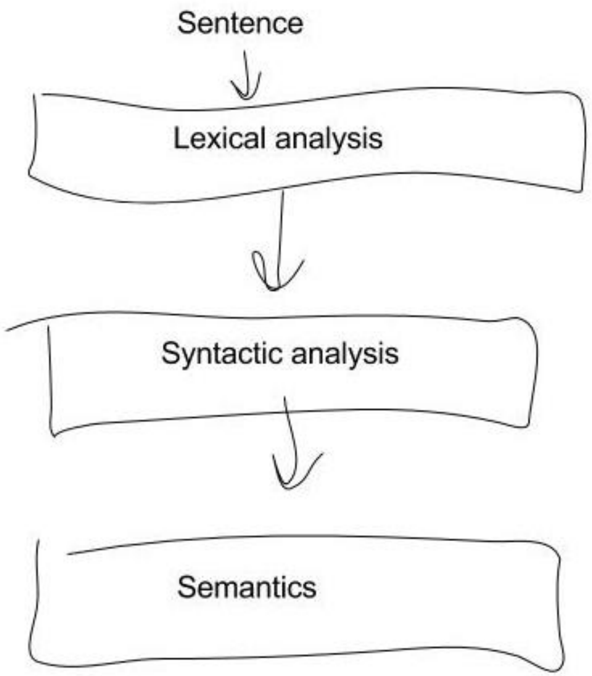
      then $< expr >.actualType = int$
      else
      then $< expr >.actualType =$ other
   2. Predicate Rule:
      $< expr >.actualType == < expr >.expectedType$
2. <expr> -> <var>
   1. Semantic Rule: $< expr >.actualType = < var >.actualType$
   2. Predicate Rule: $< expr >.actualType == < expr >.expectedType$
3. <var> -> x | y | z
   1. Semantic Rule: $< var >.actualType = symbolTableLookup(< var >.lexeme)$



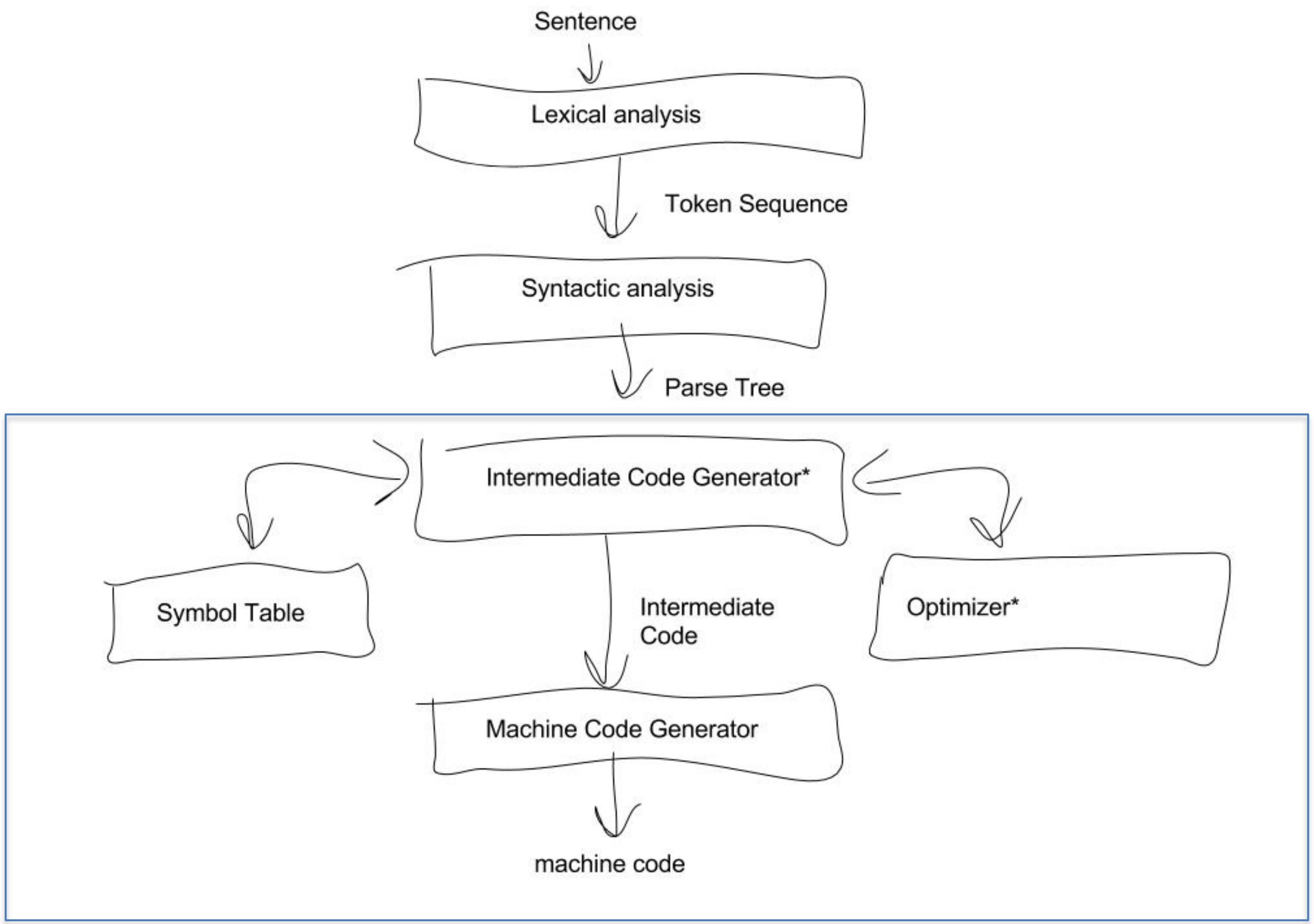
Dynamic Semantics

- After lexical and syntactic analysis, semantic analysis is performed
  - Application of *meaning* to an input sentence

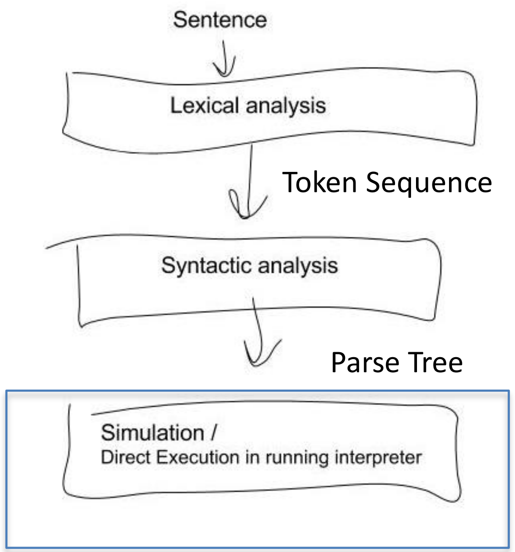


Semantics: Compiler vs Interpreter

Compiler



Interpreter



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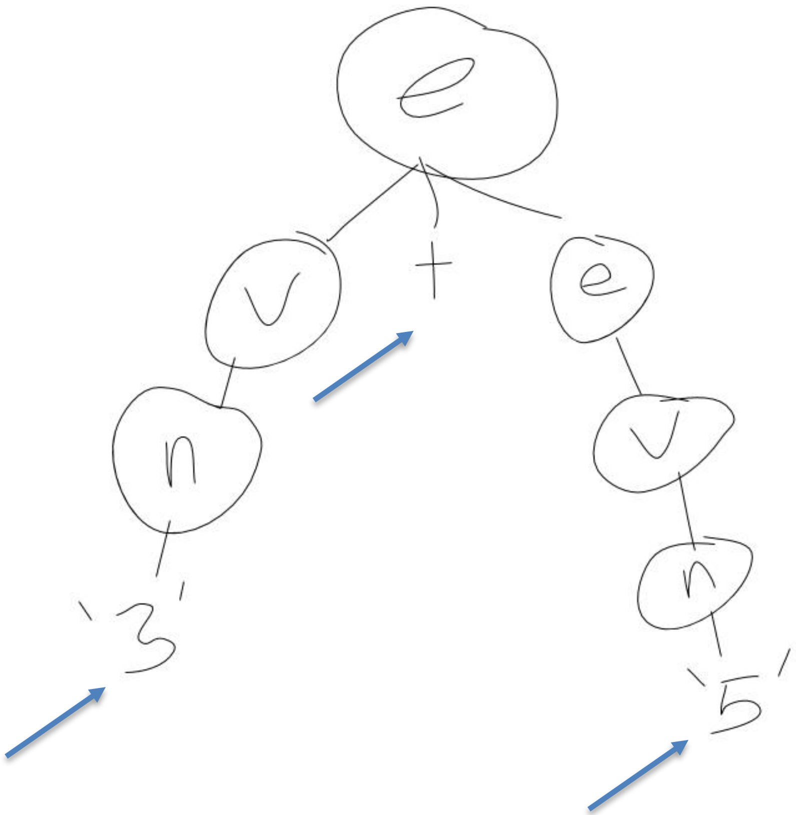
Semantics: from lexemes to abstractions

- In most languages
  - Each lexeme, syntactic unit, of a language has intrinsic meaning (semantics)
  - This semantics of an input sentence is generally determined in terms of the semantics of the lexemes of the input.
    - But How?
  - The application of semantics is driven by the BNF productions. This also implies that all characteristics of a language not specified by the BNF, must be specified in the application of semantics.
  - It is intuitive that semantics are applied based on BNF. Meaning is applied to the lexemes directly, and meaning is assigned to non-terminal constructs in terms of the meaning of its constituents
    - Meaning is propagated up the parse tree



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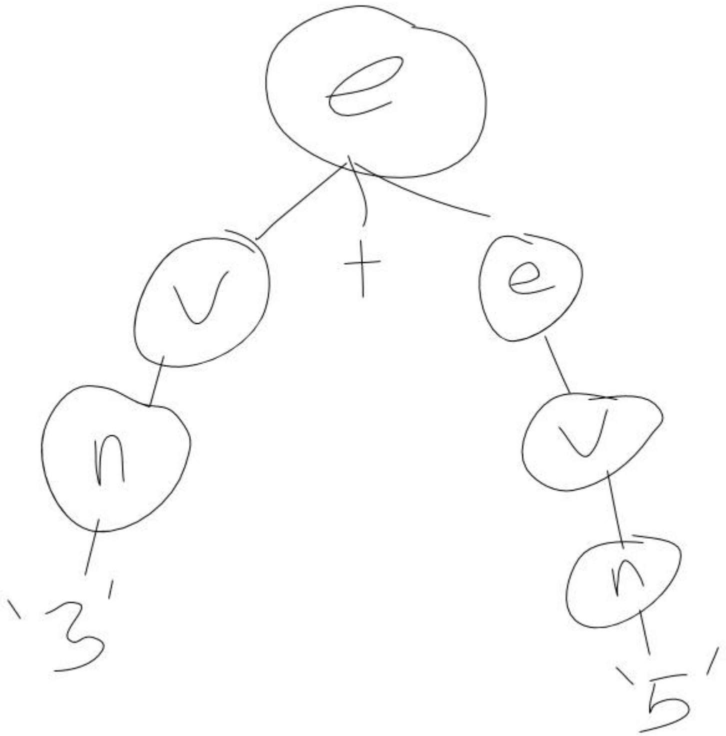
Semantics Example

- What is the meaning of “3 + 5”?
  - What is the meaning of “3” ?
    - Lexeme “3” is the 3 symbol.
    - Semantics of “3”: the number 3
    - Semantics in the context of
      Computer PL: binary rep of 3
  - What is the meaning of “+” ?
    - Lexeme “+” is the plus symbol.
    - Semantics of “+”: addition operation
    - Semantics in the context of
      Computer PL: a specific ALU operation
  - What is the meaning of “5” ?



Introduction

- In previous chapters, we discussed semantics from an informal, or descriptive, point of view
  - Historically, this has been the usual approach
- There is a need for a more mathematical description of the behavior of programs and programming languages, to make the definition of a language so precise that:
  - Programs can be **proven** correct in a mathematical way
  - Translators can be **validated** to produce exactly the behavior described in the language definition

Introduction (cont'd.)

- Developing such a mathematical system aids the designer in discovering inconsistencies and ambiguities
- There is no single accepted method for formally defining semantics
- Several methods differ in the formalisms used and the kinds of intended applications
- Formal semantic descriptions are more often supplied after the fact, and only for a portion of a language

Introduction (cont'd.)

- Formal methods have begun to be used as part of the specification of complex software projects, including language translators
- Three principal methods to describe semantics formally:
  - Operational semantics
  - Denotational semantics
  - Axiomatic semantics

Semantic Specification

- Semantic Specification determines how meaning is applied to a sentence of a language
  - A universally standardized form of semantic specification does not exist, but there are 3 general categories
    - Operational Semantics: describes the semantics of a language in terms of the state of the underlying machine
    - Denotational Semantics: describes the semantics of a language in terms of functions defined on programs and program constructs
    - Axiomatic Semantics: Uses mathematical logic to formalize characteristics of a program.

- Properties of a good semantic specification
  - It must be complete. Each input program that abides by the syntax should have appropriate semantics as defined by the specification
  - It must be consistent. Each input program must not have two conflicting semantics.

Introduction (cont'd.)

- **Operational semantics**:
  - Defines a language by describing its actions in terms of the operators of an actual or hypothetical machine
  - Requires that the operations of the machine used in the description are also precisely defined
  - A mathematical model called a "reduction machine" is often used for this purpose (similar in spirit to the notion of a Turing machine)

Introduction (cont'd.)

- **Denotational semantics**:
  - Uses mathematical functions on programs and program components to specify semantics
  - Programs are translated into functions about which properties can be proved using standard mathematical theory of functions

Introduction (cont'd.)

- **Axiomatic semantics**:
  - Applies mathematical logic to language definition
  - Assertions, or predicates, are used to describe desired outcomes and initial assumptions for program
  - Language constructs are associated with **predicate transforms** to create new assertions out of old ones
  - Transformers can be used to prove that the desired outcome follows from the initial conditions
  - Is a method aimed specifically at correctness proofs

Introduction (cont'd.)

- All these methods are syntax-directed
  - Semantic definitions are based on a context-free grammar or Backus-Naur Form (BNF) rules
- Formal semantics must then define all properties of a language that are not specified by the BNF
  - Includes static properties such as static types and declaration before use
- Formal methods can describe both static and dynamic properties
- We will view semantics as everything not specified by the BNF

Introduction (cont'd.)

- Two properties of a specification are essential:
  - Must be **complete**: every correct, terminating program must have associated semantics given by the rules
  - Must be **consistent**: the same program cannot be given two different, conflicting semantics
- Additionally, it is advantageous for the semantics to be minimal, or **independent**
  - No rule is derivable from the other rules

Introduction (cont'd.)

- Formal specifications written in operational or denotational style have an additional useful property:
  - They can be translated relatively easily into working programs in a language suitable for prototyping, such as Prolog, ML, or Haskell

Operational Semantics

- Goal: Describe semantics by specifying effects on underlying machine.
- Semantic rules are often presented in the form of reduction or logical rules
- Observations
  - The state underlying a machine has lots of details / is complex. This approach may not be practical.
  - Rather than tracking the state of a machine at a low level, this approach can be applied at an intermediate level of the computing abstraction.
    - However, this makes operational semantics difficult to formalize as the machine truly depends upon its lower level representation.

A Sample Small Language

- The basic sample language to be used is a version of the integer expression language used in Ch. 6
- BNF rules for this language:

expr → *expr* ‘+’ *term* | *expr* ‘−’ *term* | *term*
term → *term* ‘*’ *factor* | *factor*
factor → ‘(’*expr* ‘)’ | *number*
number → *number digit* | *digit*
digit → ‘0’ | ‘1’ | ‘2’ | ‘3’ | ‘4’ | ‘5’ | ‘6’ | ‘7’ | ‘8’ | ‘9’

Figure 12.1 Basic sample language

A Sample Small Language (cont'd.)

- This results in simple semantics:
  - The value of an expression is a complete representation of its meaning: 2 + 3 * 4 means 14
- Complexity will now be added to this language in stages
- In the first stage, we add variables, statements, and assignments
  - A program is a list of statements separated by semicolons
  - A statement is an assignment of an expression to an identifier

A Sample Small Language (cont'd.)

factor → '(' *expr* ')' | *number* | *identifier*

program → *stmt-list*

stmt-list → *stmt* ';' *stmt-list* | *stmt*

stmt → *identifier* ':=' *expr*

identifier → *identifier letter* | *letter*

letter → 'a' | 'b' | 'c' | . . . | 'z'

Figure 12.2 First extension of the sample language

A Sample Small Language (cont'd.)

- Semantics are now represented by a set of values corresponding to identifiers whose values have been defined, or bound, by assignments
- Example:

```
a := 2+3;
b := a*4;
a := b-5
```

  - Results in bindings `b=20` and `a=15` when it finishes
  - Set of values representing the semantics of the program is `{a=15, b=20}`

A Sample Small Language (cont'd.)

- Such a set is essentially a function from identifiers to integer values, with all unassigned identifiers having a value undefined
  - This function is called an **environment**, denoted by:

$$Env\text{: Identifier} \rightarrow \text{Integer} \cup \{\text{undef}\}$$

- Note that th … example program can be defined as:

$$Env(I) = \begin{cases} 15 \text{ if } I = a \\ 20 \text{ if } I = b \\ \text{undef otherwise} \end{cases}$$

A Sample Small Language (cont'd.)

- The operation of looking up the value of an identifier `I` in an environment `Env` is `Env(I)`
- **Empty environment** is denoted by $\mathtt{Env}_0$

$$Env_0(I) = \text{undef for all } I$$

- An environment as defined here incorporates both the symbol table and state functions
- Such environments:
  - Do not allow pointer values
  - Do not include scope information
  - Do not permit aliases

A Sample Small Language (cont'd.)

- For this view of the semantics of a program represented by a resulting final environment:
  - Consistency: we cannot derive two different final environments for the same program
  - Completeness: we must be able to derive a final environment for every correct, terminating program
- We now add `if` and `while` control statements
  - Syntax of the `if` and `while` statements borrows the Algol68 convention of writing reserved words backward, instead of `begin` and `end` blocks

A Sample Small Language (cont'd.)

stmt → *assign-stmt* | *if-stmt* | *while-stmt*
assign-stmt → *identifier* ':=' *expr*
if-stmt → 'if' *expr* 'then' *stmt-list* 'else' *stmt-list* 'fi'
while-stmt → 'while' *expr* 'do' *stmt-list* 'od'

Figure 12.3 Second extension of the sample language

A Sample Small Language (cont'd.)

- Meaning of an `if-stmt`:
  - `expr` is evaluated in the current environment
  - If it evaluates to an integer greater than 0, then `stmt-list` after `then` is executed
  - If not, `stmt-list` after the `else` is executed
- Meaning of a `while-stmt`:
  - As long as `expr` evaluates to a quantity greater than 0, `stmt-list` is repeatedly executed and `expr` is reevaluated
- Note that these semantics are nonstandard!

A Sample Small Language (cont'd.)

- Example program in this language:

```
n := 0 - 5;
if n then i := n else i := 0 - n fi;
fact := 1;
while i do
  fact := fact * i;
  i := i - 1
od
```

- Semantics are given by the final environment:

$$\{n = -5,\ i = 0,\ \text{fact} = 120\}$$

A Sample Small Language (cont'd.)

- Difficult to provide semantics for loop constructs
  - We will not always give a complete solution
- Formal semantic methods often use a simplified version of syntax from that given
- An ambiguous grammar can be used to define semantics because:
  - Parsing step is assumed to have already taken place
  - Semantics are defined only for syntactically correct constructs
- Nonterminal symbols can be replaced by single letters

A Sample Small Language (cont'd.)

- Nonterminal symbols can be replaced by single letters
  - May be thought to represent strings of tokens or nodes in a parse tree
- Such a syntactic specification is sometimes called an **abstract syntax**

A Sample Small Language (cont'd.)

- Abstract syntax for our sample language:

$$
\begin{aligned}
P &\rightarrow L \\
L &\rightarrow L_1 \text{ ';' } L_2 \mid S \\
S &\rightarrow I \text{ ':=' } E \mid \text{'if' } E \text{ 'then' } L_1, \text{ 'else' } L_2 \text{ 'fi'} \\
&\quad \mid \text{'while' } E \text{ 'do' } L \text{ 'od'} \\
E &\rightarrow E_1 \text{ '+' } E_2 \mid E_1 \text{ '–' } E_2 \mid E_1 \text{ '*' } E_2 \\
&\quad \mid \text{'(' } E_1 \text{ ')'} \mid N \\
N &\rightarrow N_1 \, D \mid D \\
D &\rightarrow \text{'0'} \mid \text{'1'} \mid \ldots \mid \text{'9'} \\
I &\rightarrow I_1 \, A \mid A \\
A &\rightarrow \text{'}a\text{'} \mid \text{'}b\text{'} \mid \ldots \mid \text{'}z\text{'}
\end{aligned}
$$

P : Program
L : Statement-list
S : Statement
E : Expression
N : Number
D : Digit
I : Identifier
A : Letter

A Sample Small Language (cont'd.)

- To define the semantics of each symbol, we define the semantics of each right-hand side of the abstract syntax rules in terms of the semantics of their parts
  - Thus, syntax-directed semantic definitions are recursive in nature
- Tokens in the grammar are enclosed in quotation marks

Operational Semantics

- Operational semantics specify how an arbitrary program is to be executed on a machine whose operation is completely known
- **Definitional interpreters** or **compilers**: translators for the language written in the machine code of the chosen machine
- Operational semantics can define the behavior of programs in terms of an **abstract machine**

Figure 12-4 Three parts of an abstract machine

Operational Semantics (cont'd.)

- **Reduction machine**: an abstract machine whose control operates directly on a program to reduce it to its semantic "value"
- Example: reduction of the expression `(3+4)*5`

```
(3 + 4) * 5 => (7) * 5      — 3 and 4 are added to get 7
            => 7 * 5        — the parentheses around 7 are dropped
            => 35           — 7 and 5 are multiplied to get 35
```

- To specify … **rules** that specify how the control reduces constructs of the language to a value

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Logical Inference Rules

- Inference rules in logic are written in the form:

$$\frac{\text{premise}}{\text{conclusion}}$$

  – If the premise is true, the conclusion is also true

- Inference rule for the commutative property of addition:

$$\frac{a + b = c}{b + a = c}$$

- Inference rules are used to express the basic rules of prepositional and predicate calculus:

$$\frac{a \to b,\ b \to c}{a \to c}$$

Logical Inference Rules (cont'd.)

- **Axioms**: inference rules with no premise
  - They are always true
  - Example:

$$a + 0 = a$$

  - Axioms can be written as an inference rule with an empty premise:

$$\frac{}{a + 0 = a}$$

  - Or without the … e:

$$a + 0 = a$$

Reduction Rules
for Integer Arithmetic Expressions

- **Structured operational semantics**: the notational form for writing reduction rules that we will use
- Semantics rules are based on the abstract syntax for expressions:

$$E \rightarrow E_1 \text{ '+' } E_2 \mid E_1 \text{ '−' } E_2 \mid E_1 \text{ '*' } E_2 \mid \text{ '(' } E_1 \text{ ')'}$$
$$N \rightarrow N_1\ D \mid D$$
$$D \rightarrow \text{'0'} \mid \text{'1'} \mid \ldots \mid \text{'9'}$$

- The notation $E \Rightarrow E_1$ states that expression `E` reduces to expression `E1` by some reduction rule

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Reduction Rules for Expressions

1. Collect all rules for reducing digits to values in this one rule
   – All are axioms

‘0’ => 0
‘1’ => 1
‘2’ => 2
‘3’ => 3
‘4’ => 4
‘5’ => 5
‘6’ => 6
‘7’ => 7
‘8’ => 8
‘9’ => 9

Reduction Rules for Expressions (cont'd.)

2. Collect all rules for reducing numbers to values in this one rule
   - All are axioms

V '0' => $10 * V$
V '1' => $10 * V + 1$
V '2' => $10 * V + 2$
V '3' => $10 * V + 3$
V '4' => $10 * V + 4$
V '5' => $10 * V + 5$
V '6' => $10 * V + 6$
V '7' => $10 * V + 7$
V '8' => $10 * V + 8$
V '9' => $10 * V + 9$

Reduction Rules for Expressions (cont'd.)

3. V_1 '+' $V_2 \Rightarrow V_1 + V_2$
4. V_1 '−' $V_2 \Rightarrow V_1 - V_2$
5. V_1 '*' $V_2 \Rightarrow V_1 * V_2$
6. '(' V ')' $\Rightarrow V$
7. $$\frac{E \Rightarrow E_1}{E \text{ '+' } E_2 \Rightarrow E_1 \text{ '+' } E_2}$$
8. $$\frac{E \Rightarrow E_1}{E \text{ '−' } E_2 \Rightarrow E_1 \text{ '−' } E_2}$$
9. $$\frac{E \Rightarrow E_1}{E \text{ '*' } E_2 \Rightarrow E_1 \text{ '*' } E_2}$$

10. $$\frac{E \Rightarrow E_1}{V \text{ '+' } E \Rightarrow V \text{ '+' } E_1}$$
11. $$\frac{E \Rightarrow E_1}{E \text{ '−' } E \Rightarrow V \text{ '−' } E_1}$$
12. $$\frac{E \Rightarrow E_1}{V \text{ '*' } E \Rightarrow V \text{ '*' } E_1}$$
13. $$\frac{E \Rightarrow E_1}{\text{'('}\ E \text{ ')'} \Rightarrow \text{'('}\ E_1 \text{ ')'}}$$
14. $$\frac{E \Rightarrow E_1,\ E_1 \Rightarrow E_2}{E \Rightarrow E_2}$$

Reduction Rules for Expressions (cont'd.)

- Rules 1 through 6 are all axioms
- Rules 1 and 2 express the reduction of digits and numbers to values
  - **Character** '0' (a syntactic entity) reduces to the **value** 0 (a semantic entity)
- Rules 3 to 5 allow an expression consisting of two values and an operator symbol to be reduced to a value by applying the appropriate operation whose symbol appears in the expression
- Rule 6 says parentheses around an expression can be dropped

Reduction Rules for Expressions (cont'd.)

- The rest of the reduction rules are inferences that allow the reduction machine to combine separate reductions together to achieve further reductions
- Rule 14 expresses the general fact that reductions can be performed stepwise (sometimes called the **transitivity rule** for reductions)

Reduction Rules for Expressions (cont'd.)

- Applying these reduction rules to the expression:

$$2 * (3 + 4) - 5.$$

- First reduce the expression: `3 + 4`:

'3' '+' '4'	=> 3 '+' '4'	(Rules 1 and 7)
	=> 3 '+' 4	(Rules 1 and 10)
	=> 3 + 4 = 7	(Rule 3)

- Thus, by rule 14, we have: '3' '+' '4' => 7.

Reduction Rules for Expressions (cont'd.)

- Continuing:

'(' '3' '+' '4' ')'	=> '(' 7 ')'	(Rule 13)
	=> 7	(Rule 6)

- Now reduce the expression `2*(3+4)` as follows:

'2' '*' '(' '3' '+' '4' ')'	=> 2 '*' '(' '3' '+' '4' ')'	(Rules 1 and 9)
	=> 2 '*' 7	(Rule 12)
	=> 2 * 7 = 14	(Rule 5)

- And finally:

'2' '*' '(' '3' '+' '4' ')' '−' '5'	=> 14 '−' '5'	(Rules 1 and 8)
	=> 14 '−' 5	(Rule 11)
	=> 14 − 5 = 9	(Rule 4)

Environments and Assignment

- Abstract syntax for our sample language:

$$
\begin{aligned}
P &\to L \\
L &\to L_1 \text{ ‘;’ } L_2 \mid S \\
S &\to I \text{ ‘:=’ } E \mid \text{‘if’ } E \text{ ‘then’ } L_1, \text{ ‘else’ } L_2 \text{ ‘fi’} \\
&\quad \mid \text{‘while’ } E \text{ ‘do’ } L \text{ ‘od’} \\
E &\to E_1 \text{ ‘+’ } E_2 \mid E_1 \text{ ‘−’ } E_2 \mid E_1 \text{ ‘*’ } E_2 \\
&\quad \mid \text{‘(’ } E_1 \text{ ‘)’} \mid N \\
N &\to N_1\, D \mid D \\
D &\to \text{‘0’} \mid \text{‘1’} \mid \ldots \mid \text{‘9’} \\
I &\to I_1\, A \mid A \\
A &\to \text{‘}a\text{’} \mid \text{‘}b\text{’} \mid \ldots \mid \text{‘}z\text{’}
\end{aligned}
$$

P : Program
L : Statement-list
S : Statement
E : Expression
N : Number
D : Digit
I : Identifier
A : Letter

Environments and Assignment (cont'd.)

- We want to extend the operational semantics to include environments and assignments
- Must include the effect of assignments on the storage of the abstract machine
- Our view of storage: an environment that is a function from identifiers to integer values (including the undefined value):

$$Env\text{: Identifier} \rightarrow \text{Integer} \cup \{\text{undef}\}$$

- The notat[illegible]ession `E` is evaluated in the presence of environment `Env`

$$<E \mid Env>$$

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Environments and Assignment (cont'd.)

- Now our reduction rules change to include environments
- Example: rule 7 with environments becomes:

$$\frac{<E \mid Env> \; => \; <E_1 \mid Env>}{<E \text{ '+' } E_2 \mid Env> \; => \; <E_1 \text{ '+' } E_2 \mid Env>}$$

  – This states that if `E` reduces to `E1` in the presence of `Env`, then `E '+' E2` reduces to `E1 '+' E2` in the same environment

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Environments and Assignment (cont'd.)

- The one case of evaluation that explicitly involves the environment is when an expression is an identifier `I`, giving a new rule:

15.

$$\frac{Env(I) = V}{<I \mid Env> \; => \; <V \mid Env>}$$

This states th is `V` in `Env`, then `I` reduces to `V` in the presence of `Env`

- Next, we add assignment statements and statement sequences to the reduction rules

Environments and Assignment (cont'd.)

- Statements must reduce to environments instead of integer values, since they create and change environments, giving this rule:

16. $<I \text{ ':=' } V \mid Env> \Rightarrow Env \ \& \ \{I = V\}$

    This states that the assignment of the value `V` to `I` in `Env` reduces to a new environment where `I` is equal to `V`

- Reduction of expressions within assignments uses this rule:

17.

$$\frac{<E \mid Env> \Rightarrow <E_1 \mid Env>}{<I \text{ ':=' } E \mid Env> \Rightarrow <I \text{ ':=' } E_1 \mid Env>}$$

Environments and Assignment (cont'd.)

- A statement sequence reduces to an environment formed by accumulating the effect of each assignment, giving this rule:

18.

$$\frac{<S \mid Env> \;=>\; Env_1}{<S \text{ ';' } L \mid Env> \;=>\; <L \mid Env_1>}$$

- Finally, a program is a statement sequence with no prior environment, giving this rule:

19.

$$L \;=>\; <L \mid Env_0>$$

It reduces to the effect it has on the empty starting environment

Environments and Assignment (cont'd.)

- Rules for reducing identifier expressions are completely analogous to those for reducing numbers
- Sample program to be reduced to an environment:

```
a := 2+3;
b := a*4;
a := b-5
```

- To simplify the reduction, we will suppress the use of quotes to differentiate between syntactic and semantic entities

Environments and Assignment (cont'd.)

- First, by rule 19, we have:

$$a := 2 + 3;\ b := a * 4;\ a := b - 5 \Rightarrow$$
$$<a := 2 + 3;\ b := a * 4;\ a := b - 5 \mid Env_0>$$

- Also, by rules 3, 17, and 16:

$$<a := 2 + 3 \mid Env_0> \Rightarrow$$
$$<a := 5 \mid Env_0> \Rightarrow$$
$$Env_0\ \&\ \{a = 5\} = \{a = 5\}$$

- Then by rule 18:

$$<a := 2 + 3;\ b := a * 4;\ a := b - 5 \mid Env_0> \Rightarrow$$
$$<b := a * 4;\ a := b - 5 \mid \{a = 5\}>$$

Environments and Assignment (cont'd.)

- Similarly, by rules 15, 9, 5, 17, and 16:

$$<b := a * 4 \mid \{a = 5\}> \; => \; <b := 5 * 4 \mid \{a = 5\}> \; =>$$
$$<b := 20 \mid \{a = 5\}> \; => \; \{a = 5\} \; \& \; \{b = 20\} = \{a = 5, b = 20\}$$

- Then by rule 18 :

$$<b := a * 4; a := b - 5 \mid \{a = 5\}> \; =>$$
$$<a := b - 5 \mid \{a = 5, b = 20\}>$$

- Finally, by a similar application of rules:

$$<a := b - 5 \mid \{a = 5, b = 20\}> \; =>$$
$$<a := 20 - 5 \mid \{a = 5, b = 20\}> \; =>$$
$$<a := 15 \mid \{a = 5, b = 20\}> \; =>$$
$$\{a = 5, b = 20\} \; \& \; \{a = 15, b = 20\}$$

Control

- Next we add if and while statements, with this abstract syntax:

$$S \to \text{`if'}\ E\ \text{`then'}\ L_1\ \text{`else'}\ L_2\ \text{`fi'}$$
$$\mid\ \text{`while'}\ E\ \text{`do'}\ L\ \text{`od'}$$

- Reduction rules for if statements include:

20.

$$\frac{<E \mid Env> \Rightarrow <E_1 \mid Env>}{<\text{`if'}\ E\ \text{`then'}\ L_1\ \text{`else'}\ L_2\ \text{`fi'} \mid Env> \Rightarrow <\text{`if'}\ E_1\ \text{`then'}\ L_1\ \text{`else'}\ L_2\ \text{`fi'} \mid Env>}$$

Control (cont'd.)

21. $$\frac{V > 0}{<\text{‘if’ } V \text{ ‘then’ } L_1 \text{ ‘else’ } L_2 \text{ ‘fi’} \mid Env> \; => \; <L_1 \mid Env>}$$

22. $$\frac{V \leq 0}{<\text{‘if’ } V \text{ ‘then’ } L_1 \text{ ‘else’ } L_2 \text{ ‘fi’} \mid Env> \; => \; <L_2 \mid Env>}$$

- Reduction rules for while statements include:

23. $$\frac{<E \mid Env> \; => \; <V \mid Env>, \; V \leq 0}{<\text{‘while’ } E \text{ ‘do’ } L \text{ ‘od’} \mid Env> \; => \; Env}$$

24. $$\frac{<E \mid Env> \; => \; <V \mid Env>, \; V > 0}{<\text{‘while’ } E \text{ ‘do’ } L \text{ ‘od’} \mid Env \; => \; < L; \text{ ‘while’ } E \text{ ‘do’ } L \text{ ‘od’} \mid Env>}$$

Implementing Operational Semantics in a Programming Language

- It is possible to implement operational semantic rules directly as a program to get an **executable specification**
- This is useful for two reasons:
  - Allows us to construct a language interpreter directly from a formal specification
  - Allows us to check the correctness of the specification by testing the resulting interpreter
- A possible Prolog implementation for the reduction rules of our sample language will be used

Implementing Operational Semantics in a Programming Language (cont'd.)

- Example: `3*(4+5)` in Prolog:

```
times(3,plus(4,5))
```

- Example: this program:

```
a := 2+3;
b := a*4;
a := b-5
```

  – Can be r… n Prolog as:

```
seq(assign(a,plus(2,3)),
    seq(assign(b,times(a,4)),assign(a,sub(b,5))))
```

- This is act… necessary…

Implementing Operational Semantics in a Programming Language (cont'd.)

- We can write reduction rules (ignoring environment rules for the moment)
- A general reduction rule for expressions:

```
reduce(X,Y) :- ...
```

  - Where `X` is any arithmetic expression (in abstract syntax) and `Y` is the result of a single reduction step applied to `X`
- Example:
  - Rule 3 can be written as:

```
reduce(plus(V1,V2),R) :-
        integer(V1), integer(V2), !, R is V1 + V2
```

Implementing Operational Semantics in a Programming Language (cont'd.)

- Rule 7 becomes:

```
reduce(plus(E,E2),plus(E1,E2)) :- reduce(E,E1)
```

- Rule 10 becomes:

```
reduce(plus(V,E),plus(V,E1)) :-
        integer(V), !, reduce(E,E1)
```

- Rule 14 presents a problem if written as:

```
reduce(E,E2) :- reduce(E,E1), reduce(E1,E2)
```

  – Infinite recursive loops will result

- Instead, write rule 14 as two rules:

```
reduce_all(V,V) :- integer(V), !.
reduce_all(E,E2) :- reduce(E,E1), reduce_all(E1,E2)
```

Implementing Operational Semantics in a Programming Language (cont'd.)

- Now extend to environments and control: a pair `<E|Env>` can be thought of as a configuration and written in Prolog as `config(E,Env)`
- Rule 15 then becomes:

```
reduce(config(I,Env),config(V,Env))  :-
                              atom(I), !, lookup(Env, I, V)
```

  – Where `atom(I)` tests for a variable and `lookup` operation finds values in an environment

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Implementing Operational Semantics in a Programming Language (cont'd.)

- Rule 16 becomes:

```
reduce(config(assign(I,V),Env),Env1) :-
      integer(V), !, update(Env, value(I,V), Env1)
```

  – Where `update` inserts the new value `V` for `I` into `Env`, yielding `Env1`

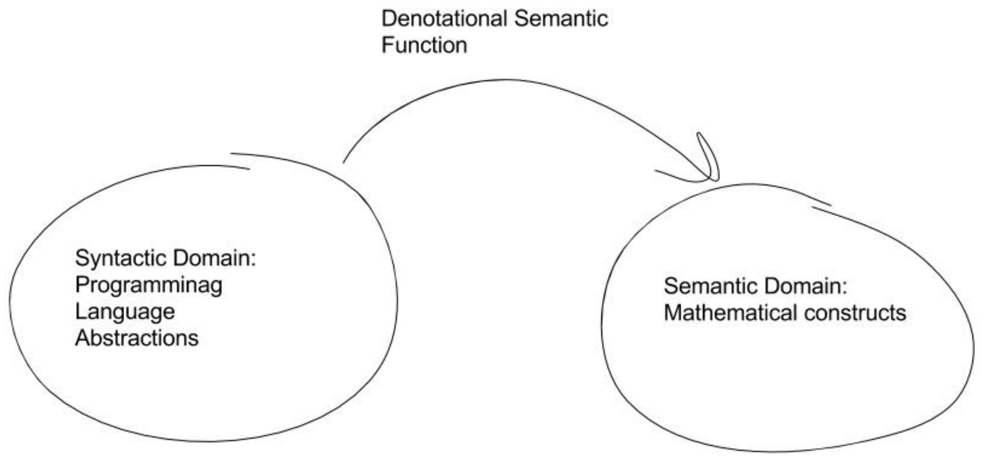
- Any dictionary structure for which `lookup` and `update` can be defined can be used to represent an environment in this code

Denotational Semantics

- Specifies semantics in terms of functions from programs and program constructs to semantics

- Observations:
  - Formal specification based on recursive function theory
  - Most rigorous
  - Most widely used

- Basic Idea
  - Define functions that map programming constructs to mathematical constructs. If we can formalize the semantics using mathematical constructs, we can then define a formal semantics for a language



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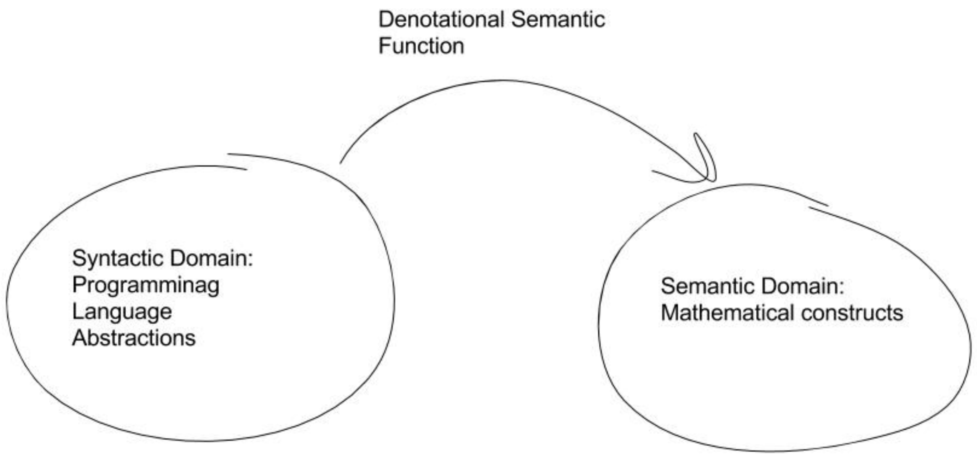
Denotational Semantics

- A denotational semantics consists of
  1. A syntactic domain: grammar productions
  2. Semantic domain: sets on which the semantic functions take their values
  3. Semantic functions: mapping from productions to values



Denotational Semantics

- Denotational semantics use functions to describe the semantics of a programming language
  - A function associates semantic values to syntactically correct constructs
- Example: a function that maps an integer arithmetic expression to its value:

$$Val : \text{Expression} \rightarrow \text{Integer}$$

  - **Syntactic domain**: domain of a semantic function
  - **Semantic domain**: range of a semantic function, which is a mathematical structure

Denotational Semantics (cont'd.)

- Example: `val(2+3*4) = 14`
  - Set of integers is the semantic domain
  - `val` maps the syntactic construct `2+3*4` to the semantic value `14`; it **denotes** the value `14`
- A program can be viewed as something that receives input and produces output
- Its semantics can be represented by a function:

$$P : \text{Program} \rightarrow (\text{Input} \rightarrow \text{Output})$$

  - Semantic d to output
  - Semantic value is a function

Denotational Semantics (cont'd.)

- Since semantic domains are often functional domains, and values of semantic functions will be functions themselves, we will assume the symbol "→" is right associative and drop the parentheses:

- Three par $P : \text{Program} \rightarrow \text{Input} \rightarrow \text{Output}$ tion of a program:
  - Definition of the **syntactic domains**
  - Definition of the **semantic domains**
  - Definition of the semantic functions themselves (sometimes called **valuation functions**)

Syntactic Domains

- **Syntactic domains**:
  - Are defined in denotational definition using notation similar to abstract syntax
  - Are viewed as sets of syntax trees whose structure is given by grammar rules that recursively define elements of the set
- Example:

D: Digit

N: Number

$N \rightarrow N\ D \mid D$

$D \rightarrow$ '0' | '1' | ... | '9'

Semantic Domains

- **Semantic domains**: sets in which semantic functions take their values
  - Like syntactic domains but may also have additional mathematical structure, depending on use
- Example: integers have arithmetic operations
- Such domains are **algebras**, which are specified by listing their functions and properties
  - Denotational definition of semantic domains lists the sets and operations but usually omits the properties of the operations

Semantic Domains (cont'd.)

- Domains sometimes need special mathematical structures that are the subject of **domain theory**
  - Term domain is sometimes reserved for an algebra with the structure of a complete partial order
  - This structure is needed to define the semantics of recursive functions and loops
- Example: semantic domain of the integers:

Domain v: Integer $= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Operations

$$+ : \text{Integer} \times \text{Integer} \to \text{Integer}$$
$$- : \text{Integer} \times \text{Integer} \to \text{Integer}$$
$$* : \text{Integer} \times \text{Integer} \to \text{Integer}$$

Semantic Functions

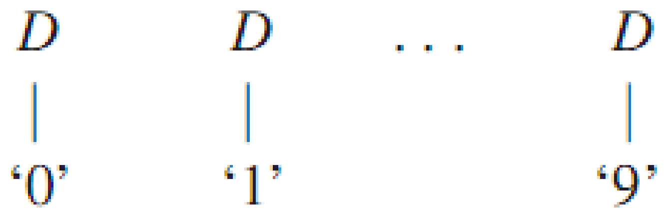
- **Semantic function**: specified for each syntactic domain
- Each function is given a different name based on its associated syntactic domain, usually with boldface letters
- Example: value function from the syntactic domain Digit to the integers:

$$D : \text{Digit} \rightarrow \text{Integer}$$

Semantic Functions (cont'd.)

- Value of a semantic function is specified recursively on the trees of syntactic domains using the structure of grammar rules
- **Semantic equation** corresponding to each grammar rule is given
- Example: grammar rule for digits:
  - Gives rise to syntax tree nodes:

$$D \rightarrow \text{'0'} \mid \text{'1'} \mid \ldots \mid \text{'9'}$$



Semantic Functions (cont'd.)

- Example (cont'd.):
  - Semantic function **D** is defined by these semantic equations representing the value of each leaf:

$$D\left(\begin{array}{c} D \\ | \\ \text{‘0’} \end{array}\right) = 0, \quad D\left(\begin{array}{c} D \\ | \\ \text{‘1’} \end{array}\right) = 1, \ldots, \quad D\left(\begin{array}{c} D \\ | \\ \text{‘9’} \end{array}\right) = 9$$

  - This notation is shorted to the following:

$$D[[\text{‘0’}]] = 0, D[[\text{‘1’}]] = 1, \ldots, D[[\text{‘9’}]] = 9$$

  - Double bra … nt is a syntactic entity consisting of a syntax tree node with the listed arguments as children

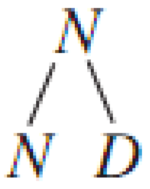
Semantic Functions (cont'd.)

- Example: semantic function from numbers to integers:
  - Is based on the syn $N : \text{Number} \to \text{Integer}$
  - And is given by these equations: $N \to N\,D \mid D$

$$N[[ND]] = 10 * N[[N]] + N[[D]]$$

$$N[[D]] = D[[D]]$$

  - Where [[N



  - And [[D]] refers to the node

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Denotational Semantics of Integer Arithmetic Expressions

Syntactic Domains

E: Expression
N: Number
D: Digit

$E \rightarrow E_1$ '+' E_2 | E_1 '−' E_2 | E_1 '*' E_2
| '(' E ')' | N
$N \rightarrow N\ D$ | D
$D \rightarrow$ '0' | 1' | . . . | '9'

Semantic Domains

Domain v: Integer = {. . ., −2, −1, 0, 1, 2, . .
Operations
+ : Integer × Integer → Integer
− : Integer × Integer → Integer
* : Integer × Integer → Integer

Semantic Functions

$\boldsymbol{E}$: Expression → Integer

$\boldsymbol{E}$[[E_1 '+' E_2]] = $\boldsymbol{E}$[[E_1]] + $\boldsymbol{E}$[[E_2]]
$\boldsymbol{E}$[[E_1 '−' E_2]] = $\boldsymbol{E}$[[E_1]] − $\boldsymbol{E}$[[E_2]]
$\boldsymbol{E}$[[E_1 '*' E_2]] = $\boldsymbol{E}$[[E_1]] * $\boldsymbol{E}$[[E_2]]
$\boldsymbol{E}$[['(' E ')']] = $\boldsymbol{E}$[[E]]
$\boldsymbol{E}$[[N]] = $\boldsymbol{N}$[[N]]

$\boldsymbol{N}$: Number → Integer

$\boldsymbol{N}$[[ND]] = 10 * $\boldsymbol{N}$[[N]] + $\boldsymbol{N}$[[D]]
$\boldsymbol{N}$[[D]] = $\boldsymbol{D}$[[D]]

$\boldsymbol{D}$: Digit → Integer

$\boldsymbol{D}$[['0']] = 0, $\boldsymbol{D}$[['1']] = 1,. . ., $\boldsymbol{D}$[['9']] = 9

Denotational Semantics of Integer Arithmetic Expressions (cont'd.)

- Using these equations to obtain the semantic value of an expression, we compute $\boldsymbol{E}[[(2+3)*4]]$ecisely,

$$\boldsymbol{E}[[\text{‘(’ ‘2’ ‘+’ ‘3’ ‘)’ ‘*’ ‘4’}]]$$

$$
\begin{aligned}
&\boldsymbol{E}[[\text{‘(’ ‘2’ ‘+’ ‘3’ ‘)’ ‘*’ ‘4’}]] \\
&\quad = \boldsymbol{E}[[\text{‘(’ ‘2’ ‘+’ ‘3’ ‘)’}]] * \boldsymbol{E}[[\text{‘4’}]] \\
&\quad = \boldsymbol{E}[[\text{‘2’ ‘+’ ‘3’}]] * \boldsymbol{N}[[\text{‘4’}]] \\
&\quad = (\boldsymbol{E}[[\text{‘2’}]] + \boldsymbol{E}[[\text{‘3’}]]) * \boldsymbol{D}[[\text{‘4’}]] \\
&\quad = (\boldsymbol{N}[[\text{‘2’}]] + \boldsymbol{N}[[\text{‘3’}]]) * 4 \\
&\quad = \boldsymbol{D}[[\text{‘2’}]] + \boldsymbol{D}[[\text{‘3’}]]) * 4 \\
&\quad = (2 + 3) * 4 = 5 * 4 = 20
\end{aligned}
$$

Environments and Assignments

- First extension to our sample language adds identifiers, assignment statements, and environments
- Environments are functions from identifiers to integers (or undefined)
- Set of environments becomes a new semantic domain:

$$\text{Domain } \mathit{Env}\text{: Environment} = \text{Identifier} \rightarrow \text{Integer} \cup \{\text{undef}\}$$

Environments and Assignments (cont'd.)

- In denotational semantics, the value `undef` is called **bottom**, from the theory of partial orders, and is denoted by the symbol $\perp$
- Semantic domains with this value are called **lifted domains** and are subscripted with the symbol $\perp$
- The initial environment defined previously can now be defined as:

$$Env_0(I) = \perp \text{ for all identifiers } I.$$

- Semantic value of an expression becomes a function from environments to integers:

$$E : \text{Expression} \rightarrow \text{Environment} \rightarrow \text{Integer}_\perp$$

Environments and Assignments (cont'd.)

- The value of an identifier is its value in the environment provided as a parameter:

$$E[[I]](Env) = Env(I)$$

- For a number, the environment is immaterial:

$$E[[N]](Env) = N[[N]]$$

- For statements and statement lists, the semantic values are functions from environments to environments
  - The "&" notation is used to add values to functions that we have used in previous sections

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Syntactic Domains

P: Program
L: Statement-list
S: Statement
E: Expression
N: Number
D: Digit
I: Identifier
A: Letter

$P \rightarrow L$
$L \rightarrow L_1$ ';' L_2 | S
$S \rightarrow I$ ':=' E
$E \rightarrow E_1$ '+' E_2 | E_1 '−' E_2 | E_1 '*' E_2
| '(' E ')' | I | N
$N \rightarrow N\,D$ | D
$D \rightarrow$ '0' | '1' |...| '9'
$I \rightarrow I\,A$ | A
$A \rightarrow$ 'a' | 'b' |...| 'z'

Figure 12.5 A denotational definition for the sample language extended with assignment statements and environments (*continues*)

Semantic Domains

Domain v: Integer = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Operations

$$+ : \text{Integer} \times \text{Integer} \to \text{Integer}$$
$$- : \text{Integer} \times \text{Integer} \to \text{Integer}$$
$$* : \text{Integer} \times \text{Integer} \to \text{Integer}$$

Domain Env: Environment = Identifier $\to$ $\text{Integer}_{\perp}$

Semantic Functions

P : Program $\to$ Environment

$$P[[L]] = L[[L]](Env_0)$$

L : Statement-list $\to$ Environment $\to$ Environment

$$L[[L_1 \text{ `;' } L_2]] = L[[L_2]] \circ L[[L_1]]$$
$$L[[S]] = S[[S]]$$

Figure 12.5 A denotational definition for the sample language extended with assignment statements and environments (*continues*)

S : Statement → Environment → Environment

$$S[[\ I\ `{:=}'\ E\ ]](Env) = Env\ \&\ \{I = E[[E]](Env)\}$$

E : Expression → Environment → $\text{Integer}_\perp$

$$E[[E_1\ `{+}'\ E_2]](Env) = E[[E_1]](Env) + E[[E_2]](Env)$$
$$E[[E_1\ `{-}'\ E_2]](Env) = E[[E_1]](Env) - E[[E_2]](Env)$$
$$E[[E_1\ `{*}'\ E_2]](Env) = E[[E_1]](Env) * E[[E_2]](Env)$$
$$E[[`(\text{'}\ E\ `)\text{'}]](Env) = E[[E]](Env)$$
$$E[[I]](Env) = Env(\ I\ )$$
$$E[[N]](Env) = N[[N]]$$

N: Number → Integer

$$N[[ND]] = 10*N[[N]] + N[[D]]$$
$$N[[D]] = D[[D]]$$

D : Digit → Integer

$$D[[`0\text{'}]] = 0, D[[`1\text{'}]] = 1, \ldots, D[[`9\text{'}]] = 9$$

Figure 12.5 A denotational definition for the sample language extended with assignment statements and environments

Denotational Semantics of Control Statements

- if and while statements have this abstract syntax:

$$
\begin{aligned}
&S\text{: Statement}\\
&S \to I \text{ ‘:=’ } E\\
&\quad\mid \text{ ‘if’ } E \text{ ‘then’ } L_1 \text{ ‘else’ } L_2 \text{ ‘fi’}\\
&\quad\mid \text{ ‘while’ } E \text{ ‘do’ } L \text{ ‘od’}
\end{aligned}
$$

- Denotational semantics is given by a function from environments to environments:

$$S : \text{Statement} \to \text{Environment} \to \text{Environment}$$

- Semantic function of the if statement:

$$
\begin{aligned}
&S[[\text{‘if’ } E \text{ ‘then’ } L_1 \text{ ‘else’ } L_2 \text{ ‘fi’}]](Env) =\\
&\qquad \text{if } E[[E]](Env) > 0 \text{ then } L[[L_1]](Env) \text{ else } L[[L_2]](Env)
\end{aligned}
$$

Denotational Semantics of Control Statements (cont'd.)

- Semantic function for the while statement is more difficult
  - Can construct a function as a set by successively extending it to a **least-fixed-point solution**, the "smallest" solution satisfying the equation
  - Here, F will be a function on the semantic domain of environments
- Must also deal with nontermination in loops by assigning the "undefined" value

$\bot$

Denotational Semantics of Control Statements (cont'd.)

- The domain of environments becomes a lifted domain:

$$\text{Environment}_{\perp} = (\text{Identifier} \rightarrow \text{Integer}_{\perp})_{\perp}$$

- Semantic func :

$$S : \text{Statement} \rightarrow \text{Environment}_{\perp} \rightarrow \text{Environment}_{\perp}$$

Implementing Denotational Semantics in a Programming Language

- We will use Haskell for a possible implementation of the denotational functions of the sample language
- Abstract syntax of expressions:

```
data Expr = Val Int | Ident String | Plus Expr Expr
                    | Minus Expr Expr | Times Expr Expr
```

- We ignore the semantics of numbers and simply let values be integers

Implementing Denotational Semantics in a Programming Language (cont'd.)

- Assume we have defined an Environment type with a lookup and update operation
- The `E` evaluation function can be defined as:

```
exprE :: Expr -> Environment -> Int
exprE (Plus e1 e2) env = (exprE e1 env) + (exprE e2 env)
exprE (Minus e1 e2) env = (exprE e1 env) - (exprE e2 env)
exprE (Times e1 e2) env = (exprE e1 env) * (exprE e2 env)
exprE (Val n) env = n
exprE (Ident a) env = lookup env a
```

Another: Denotation Semantics Example

- Example Grammar (includes int definition)
  - $E \rightarrow E_1 \text{"} + \text{"} E_2 \quad | \; E_1 \text{"} - \text{"} \; E_2 \; | \;\; E_1 \text{"}*\text{"} \; E_2 | \;\; N$
  - $N \rightarrow ND \mid D$
  - $D \rightarrow \text{"0"|"1"|"2"} | \ldots | \text{"9"}$

- Semantic Domains:
  - Domain: Integer
  - Operations:
    - $+\!: \; Integer \; X \; Integer \;\; \rightarrow Integer$
    - $-\!: Integer \; X \; Integer \;\; \rightarrow Integer$
    - $*\!: Integer \; X \; Integer \;\; \rightarrow Integer$

- Semantic Functions:
  - Eval_D: <u>D</u>igit => Integer

    Eval_D[“0”] = 0
    Eval_D[“1”] = 1
    …
  - Eval_N: <u>N</u>umber => Integer

    Eval_N[ND] = 10*Eval_N[N] + Eval_D[D]
    Eval_N[ND] = 10*Eval_N[N] + Eval_N[D]
    Eval_N[D] = Eval_D[D]
  - Eval_E: Expression => Integer

    Eval_E[$E_1 \text{"} + \text{"} E_2$] = Eval_E[$E_1$] + Eval_E[$E_2$]
    Eval_E[N] = Eval_N[N]

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Denotation Semantics: Example with runtime environment

- Environment is often formalized as a parameter to the semantic functions

- Example Grammar (includes int definition)
  - S→ $ID"="E;$
  - $E \rightarrow E_1"+"E_2 \quad | E_1"-"E_2 \; | \; E_1"*"E_2 | \; N$
  - $N \rightarrow ND \mid D$
  - $D \rightarrow "0"|"1"|"2"| \ldots |"9"$
  - ID → {[a-z]}

- Semantic Domains:
  - Domain: Set of all Enviroments
    - $Enviroment = ID \rightarrow Integer$
  - Domain: Integer
  - Operations:
    - $+\!: Integer\ X\ Integer \rightarrow Integer$
    - $-\!: Integer\ X\ Integer \rightarrow Integer$
    - $*\!: Integer\ X\ Integer \rightarrow Integer$

- Semantic Functions:
  - Eval_S$_{Env}$ [S]: S => Environment
    - Eval_S$_{Env}$ [S] = Env ∪ (ID, Eval_E$_{Env}$ [E])
  - Eval_D: Digit => Integer

    Eval_D$_{Env}$["0"] = 0

    Eval_D$_{Env}$ ["1"] = 1

    …
  - Eval_N: Number => Integer

    Eval_N$_{Env}$ [ND] = 10*Eval_N$_{Env}$ [N] + Eval_D$_{Env}$ [D]

    Eval_N$_{Env}$ [ND] = 10*Eval_N$_{Env}$ [N] + Eval_N$_{Env}$ [D]

    Eval_N$_{Env}$ [D] = Eval_D$_{Env}$ [D]
  - Eval_E: Expression => Integer

    Eval_E$_{Env}$ [$E_1"+"E_2$] = Eval_E$_{Env}$ [E_1] + Eval_E$_{Env}$ [E_2]

    Eval_E$_{Env}$ [N] = Eval_N$_{Env}$ [N]

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Axiomatic Semantics

- Formalizes semantics via mathematical logic
- Has no model for the state of the machine
- Generally used to determine algorithm correctness, or other characteristics / constraints related to an algorithm
- Observations: generally not a comprehensive specification for semantics
  - Preconditions
  - Postconditions

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Axiomatic Semantics

- **Axiomatic semantics**: define the semantics of a program, statement, or language construct by describing the effect its execution has on assertions about the data manipulated by the program
- Elements of mathematical logic are used to specify the semantics, including logical axioms
- We consider logical assertions to be statements about the behavior of the program that are true or false at any moment during execution

Axiomatic Semantics (cont'd.)

- **Preconditions**: assertions about the situation just before execution
- **Postconditions**: assertions about the situation just after execution
- Standard notation is to write the precondition inside curly brackets just before the construct and write the postcondition similarly just after the construct:

$$\{x = A\}\ \texttt{x := x + 1}\ \{x = A + 1\}$$

or

$$\{x = A\}$$
$$\texttt{x := x + 1}$$
$$\{x = A + 1\}$$

Axiomatic Semantics (cont'd.)

- Example: `x := 1 / y`
  - Semantics become:

$\{y \neq 0\}$
`x := 1 / y`
$\{x = 1/y\}$

- Such pre- and postconditions are often capable of being tested for validity during execution, as a kind of error checking
  - Conditions are usually Boolean expressions
- In C, can use the `assert.h` macro library for checking assertions

Axiomatic Semantics (cont'd.)

- An **axiomatic specification** of the semantics of the language construct `C` is of the form $\{P\}\ C\ \{Q\}$
  - Where `P` and `Q` are assertions
  - If `P` is true just before execution of `C`, then `Q` is true just after execution of `C`
- This representation of the action of `C` is not unique and may not completely specify all actions of `C`
- **Goal-oriented activity**: way to associate to C a general relation between precondition P and postcondition Q
  - Work backward from the goal to the requirements

Axiomatic Semantics (cont'd.)

- There is one precondition P that is the **most general** or **weakest** assertion with the property that
  $\{P\}\ C\ \{Q\}$
  - Called the **weakest precondition** of postcondition Q and construct C
  - Written as $wp(C,Q)$.
- Can now restate the property as

  $\{P\}\ C\ \{Q\}$ if and only if $P \rightarrow wp(C,Q)$

Axiomatic Semantics (cont'd.)

- We define the axiomatic semantics of language construct `C` as the function from assertion to a$wp(C, _)$on
  - Called a **predicate transformer**: takes a predicate as argument and returns a predicate result
  - Computes the weakest precondition from any postcondition
- Example assignment can now be restated as:

$$wp(\texttt{x := 1/y, x = 1/y}) = \{\texttt{y} \neq 0\}$$

General Properties of wp

- Predicate transformer $wp(C,Q)$ rtain properties that are true for almost all language constructs C
- **Law of the Excluded Miracle**:
  - There is nothing a construct C can do that $wp(C,\text{false}) = \text{false}$ true
- **Distributivity of Conjunction**:

$$wp(C,P \text{ and } Q) = wp(C,P) \text{ and } wp(C,Q)$$

- **Law of Mo**

$$\text{if } Q \rightarrow R \text{ then } wp(C,Q) \rightarrow wp(C,R)$$

General Properties of wp (cont'd.)

- **Distributivity of Disjunction**:

$$wp(C,P) \text{ or } wp(C,Q) \rightarrow wp(C,P \text{ or } Q)$$

- The last two properties regard implication operator "→" and "or" operator with equality if C is deterministic
- The question of determinism adds complexity
  - Care must be taken when talking about any language construct

Axiomatic Semantics of the Sample Language

- The specification of the semantics of expressions alone is not commonly included in an axiomatic specification
- Assertions in an axiomatic specificator are primarily statements about the side effects of constructs
  - They are statements involving identifiers and environments

Axiomatic Semantics of the Sample Language (cont'd.)

- Abstract syntax for which we will define the *wp* operator:

$$
\begin{aligned}
&P \rightarrow L \\
&L \rightarrow L_1 \text{ ‘;’ } L_2 \mid S \\
&S \rightarrow I \text{ ‘:=’ } E \\
&\qquad \mid \text{ ‘if’ } E \text{ ‘then’ } L_1 \text{ ‘else’ } L_2 \text{ ‘fi’} \\
&\qquad \mid \text{ ‘while’ } E \text{ ‘do’ } L \text{ ‘od’}
\end{aligned}
$$

- The first two rules do not need separate specifications
  - The *wp* operator for program `P` is the same as for its associated statement-list `L`

Axiomatic Semantics of the Sample Language (cont'd.)

- **Statement-lists**: for lists of statements separated by a semicolon, we have:

$$wp(L_1; L_2, Q) = wp(L_1, wp(L_2, Q))$$

  – The weakest precondition of a series of statements is the composition of the weakest preconditions of its parts

- **Assignment statements**: definition of *wp* is:

$$wp(I := E, Q) = Q[E/I]$$

  – $Q[E/I]$ is the assertion Q, with E replacing all free occurrences of the identifier I $Q[E/I]$

Axiomatic Semantics of the Sample Language (cont'd.)

- Recall that an identifier `I` is **free** in a logical assertion `Q` if it is not **bound** by either the existential quantifier "there exists" or the universal quantifier "for all"
- $wp(I := E, Q) = Q[E/I]$ says that for `Q` to be true after the assignment `I:=E`, whatever `Q` says about `I` must be true about `E` before the assignment is executed
- **If statements**: our semantics of the if statement state that the expression is true if it is greater than 0 and false otherwise

Axiomatic Semantics of the Sample Language (cont'd.)

- Given the if statement: $\texttt{if}\ E\ \texttt{then}\ L_1\ \texttt{else}\ L_2\ \texttt{fi}$
- The weakest precondition is defined as:

$$wp(\texttt{if}\ E\ \texttt{then}\ L_1\ \texttt{else}\ L_2\ \texttt{fi}, Q) = (E > 0 \rightarrow wp(L_1,Q))\ \text{and}\ (E \le 0 \rightarrow wp(L_2,Q))$$

- **While statements**: $\texttt{while}\ E\ \texttt{do}\ L\ \texttt{od}$ executes as long as `E>0`
- Must give an inductive definition based on the number of times the loop executes
- Let $H_i(\texttt{while}\ E\ \texttt{do}\ L\ \texttt{od}, Q)$ be a statement that the loop executes `I` times and terminates satisfying `Q`

Axiomatic Semantics of the Sample Language (cont'd.)

- Then

$$H_0(\texttt{while } E \texttt{ do } L \texttt{ od}, Q) = E \le 0 \text{ and } Q$$

- And

$$H_1(\texttt{while } E \texttt{ do } L \texttt{ od}, Q) = E > 0 \text{ and } wp(L, Q \text{ and } E \le 0)$$
$$= E > 0 \text{ and } wp(L, H_0(\texttt{while } E \texttt{ do } L \texttt{ od}, Q))$$

- Continuing, we have in general that:

$$H_{i+1}(\texttt{while } E \texttt{ do } L \texttt{ od}, Q) =$$
$$E > 0 \text{ and } wp(L, H_i(\texttt{while } E \texttt{ do } L \texttt{ od}, Q))$$

- Now we define:

$$wp(\texttt{while } E \texttt{ do } L \texttt{ od}, Q)$$
$$= \text{there exists an } i \text{ such that } H_i(\texttt{while } E \texttt{ do } L \texttt{ od}, Q)$$

Axiomatic Semantics of the Sample Language (cont'd.)

- Note that this definition of the semantics of the while requires that the loop terminates
- A non-terminating loop always has false as its weakest precondition (it can never make a postcondition true)

$$wp(\texttt{while 1 do } L \texttt{ od}, Q) = \text{false, for all } L \text{ and } Q$$

- These semantics for loops are difficult to use in the area of proving correctness of programs

Proofs of Program Correctness

- The major application of axiomatic semantics is as a tool for proving correctness of programs
- Recall that C satisfies a specification $\{P\}\ C\ \{Q\}$ded
- To prove correctne $P \to wp(C,Q)$
  1. Compute *wp* from the axiomatic semantics and general properties of *wp*
  2. Show that

$$P \to wp(C,Q)$$

Proofs of Program Correctness (cont'd.)

- To show that a while-statement is correct, we only need an **approximation** of its weakest precondition, that is some assertion `W` such that

$$W \to wp(\texttt{while} \ldots, Q).$$

- If we can show that `P→W`, we have also shown the correctness of `{P} while… {Q}`, since `P→W` and `W→wp(while…,Q)` imply that `P→wp(while…,Q)`

Proofs of Program Correctness (cont'd.)

- Given the loop `while` E `do` L `od` we need to find an assertion `W` such that these conditions are true:

$$\text{(a)} \quad W \text{ and } (E > 0) \rightarrow wp(L, W)$$

$$\text{(b)} \quad W \text{ and } (E \leq 0) \rightarrow Q$$

$$\text{(c)} \quad P \rightarrow W$$

  - Every time … to be true by condition (a)
  - When the loop terminates, (b) says `Q` must be true
  - (c) implies that `W` is the required approximation for

$$wp(\texttt{while} \ldots, Q)$$

Proofs of Program Correctness (cont'd.)

- An assertion `W` satisfying condition (a) is called a loop invariant for the loop, since a repetition of the loop leaves `W` true
  - In general, loops have many invariants `W`
  - Must find an appropriate `W` that also satisfies conditions (b) and (c)

Axiomatic Semantics

- Logical assertions (predicates) are denoted in braces
  {preconditions}
  statements
  {postconditions}

- Example
  {x > 0}
  sum = x + 1
  {sum > 1}

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Axiomatic Semantics

- Axiomatic semantic specification
  - Can we use axiomatic semantics to specify a language

- Look at {P} C {Q}
  - We can attempt to specify C via the pre condition P and the post condition Q. P -> Q. However, in general, this will not uniquely specify C.
  - However, we can use this logical framework to determine what preconditions are necessary to achieve some postcondition.

Axiomatic Semantics: Example

- Note that there are many assertions P, with property {P}C{Q}
- Example

  {P}
  sum = x + 1
  {sum > 1}

P could be x > 0, or x > 1, or x > 2, …

It is often desired to know the most general assertion or *weakest precondition* P of postcondition Q given programming construct C, such that {P}C{Q}. Also written *wp(C,Q)*

Weakest Precondition

- Example
  - What is the weakest precondition P

1. {P}
   y = x – 7
   {y < 0}

2. wp(x = x + 5 , x = 10)

3. {P}
   y = 1/x
   {y > 5}

Weakest Precondition

- Statement Lists
  - $wp(L_1;L_2, Q) = wp(L_1, wp(L_2,Q))$

- Example

  {P}
  y = x + 5
  z = y * 2
  {z < 0}

  wp(y = x+5; z = y * 2, Q) = wp(y = x + 5, wp(z = y * 2 , z<0))
  = wp(y = x + 5, y < 0)
  = x < -5

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Exercise: Try the following

1. {P}
   y = x + 5
   z = y / 2
   {z < 0}

2. {P}
   y = 5 * x
   z = y / 2
   {1 > z > 0}

Axiomatic Semantics for proof of program correctness

- Program correctness idea. Assume we have an assertion of the form {P} C {Q}. If we can show that P implies wp(C,{Q}), then we can conclude that the assertion {P} C {Q} is true.
  - Set up. Define Q to assure that program is “correct”.
  - Next, either confirm some P implies wp(C,{Q}) , or solve for P = wp(C,{Q})

- Prove the following is swap algorithm is correct.

swapXY:

```
    t = x
    x = y
    y = t
```

- Using the weakest precondition to show that {P} C {Q},

```
{x = X, y = Y}
    t = x
    x = y
    y = t
{y = X, x = Y}
```

```
wp(t=x;x=y;y=t, {y=X, x=Y}) = wp(t=x;x=y, wp( y=t, {y=X, x=Y} ) )
                            =  wp(t=x, wp(x=y,   {t = X, x=Y}))
                            =  wp(t=x, {t = X, y = Y}))
                            = {x = X, y = Y}
```

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Appendix

Operational Semantics: Simple Expressions Example

- CFG:
  - <e> -> $< n_1 > \ + < n_2 >$
  - <n> -> 0 | 1 | 2 | … | 9
- Semantic rules: as Reduction rules and logic rules
  - “0” => 0 (binary representation of zero) , string zero reduced to value zero
  - “1” => 1 (binary representation of one)
  - …
  - “9” => 9 (binary representation of nine)
  - $X_1"+"X_2$ => $\mathrm{X}_1 \ + X_2$ (addition of $\mathrm{X}_1 \ and \ X_2$) , two values combined by “+” reduces to addition of the two values
  - $\frac{n \Rightarrow n_1}{n \ \ "+"n_2 \Rightarrow n_1"+"n_2}$ if n resolves to n_1 then $n \ \ "+"n_2$ resolves to $n_1"+"n_2$
  - $\frac{n \Rightarrow n_1}{X + n_1 \Rightarrow X\ "+"n}$ if n resolves to n_1 then X $"+"n_1$ resolves to X $"+"n$

Operational Semantics: Runtime Environment Example

- To make assignments we must specify our runtime environment
  - $Env: identifiers \rightarrow values$
- CFG:
    - <s> => <v> = <e> ;
    - <e> => $< n_1 > \ + < n_2 >$
    - <n> => 0 | 1 | 2 | … | 9 | <id>
    - <id> => x | y | z
  - We can modify our previous rules to include the idea of environment.
    - $\frac{n \Rightarrow n_1}{n \ \text{"+"} n_2 \Rightarrow n_1 \text{"+"} n_2}$ if n resolves to n_1 then $n \ \text{" + "} n_2$ resolves to $n_1 \text{" + "} n_2$
    - $\frac{\{n|Env\} \Rightarrow \{n_1|Env\}}{\{n \ + n_2| \ Env\} \Rightarrow \{n_1 \text{"+"} n_2|Env\}}$ if n resolves to n_1 given runtime environment Env, then then $n \ \text{" + "} n_2$ resolves to $n_1 \text{" + "} n_2$ given runtime environment Env.

Operational Semantics: Runtime Environment Example

- Using this notation, we can use operational semantics to determine how to evaluate an identifier and how to assign values to an identifier

$$\frac{Env(id)=X}{\{id|\ Env\}\Rightarrow\{X|Env\}}$$

if id maps to X given the mapping, then id evaluates to X in the environment

$$\{id\ "="\ \mathrm{V}\ |\ \mathrm{Env}\}\quad => \quad \mathrm{Env} = \mathrm{Env}\ \cup (id, V)$$

add mapping of id to V to the environment

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